SINGULAR R-MATRICES AND DRINFELD'S COMULTIPLICATION

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ABSTRACT. We compute the R-matrix which intertwines two dimensional evaluation representations with Drinfeld comultiplication for $U_q(\widehat{sl}_2)$ [Dr2]. This R-matrix contains terms proportional to the δ-function. We construct the algebra $A(R)$ [RSTS] generated by the elements of the matrices $L^{\pm}(z)$ with relations determined by R. In the category of highest weight representations there is a Hopf algebra isomorphism between $A(R)$ and an extension $\overline{U}_q(\widehat{sl}_2)$ of Drinfeld's algebra.

1. Introduction

This note contains two results concerning the algebra $U_q(\widehat{sl}_2)$ with Drinfeld comultiplication [Dr2]. This algebra is presented in terms of current generators, and as an algebra, it is isomorphic [B] to the Drinfeld-Jimbo $U_q(\widehat{sl}_2)$ [Dr, J]. However it has a different comultiplication and therefore a different Hopf algebra structure.

The first result presented here is a direct calculation of the R-matrix which acts as an intertwiner of two dimensional evaluation representations. The second is the construction of the L-matrix algebra $A(R)$, presented in terms of relations between generators $L^{\pm}(z)$, following the construction in [RSTS], and the isomorphism between $A(R)$ and Drinfeld's algebra.

The quantum R-matrix which intertwines evaluation representations with Drinfeld comultiplication is completely determined, up to a scalar factor, from the evaluation representation of Drinfeld's generators and the action of the comultiplication on these generators, from the equation

$$\widetilde{R}(\tfrac{z}{w})(\pi_z\otimes\pi_w)\Delta(x) = (\pi_z\otimes\pi_w)\Delta'(x)\widetilde{R}(\tfrac{z}{w}), \qquad x\in U_q(\widehat{sl}_2)$$

(see section 2.3 for notation). These relations are considered here as relations between formal power series. It is shown that this implies that the R-martrix contains terms proportional to the δ-function.

This matrix is then used to construct the algebra $A(R)$, generated by elements of the triangular matrices $L^{\pm}(z)$, with relations determined by the R-matrix, of the form

$$\begin{aligned} R(\tfrac{z}{w})L_1^{\pm}(z)L_2^{\pm}(w) &= L_2^{\pm}(w)L_1^{\pm}(z)R(\tfrac{z}{w}),\\ R(\tfrac{z}{w}q^{-c})L_1^{+}(z)L_2^{-}(w) &= L_2^{-}(w)L_1^{+}(z)R(\tfrac{z}{w}q^{c}) \end{aligned}$$

(see section 3 for the precise definition), in a construction following the method of [RTF, RSTS]. This algebra has a natural Hopf algebra structure, and the intertwining relation of the R-matrix follows easily from the Yang-Baxter relation for R. We construct the Hopf algebra isomorphism between $A(R)$ and an extended version of Drinfeld's algebra, $\overline{U}_q(\widehat{sl}_2)$.

In the final section the Poisson limit $q \to 1$ of the quantum algebra $A(R)$ is computed. The relations in this Poisson algebra are determined by the classical r-matrix.

It is straightforward to generalize the construction presented here to the case of $U_q(\widehat{sl}_n)$ [K]. However most of the new features of the construction are already seen in the case $n = 2$.

2. Drinfeld's "new realization" of $U_q(\widehat{sl}_2)$

2.1. **Generators and relations.** Consider the Drinfeld realization of the algebra $U_q(\widehat{sl}_2)$ [Dr2] generated by

$$q^{\pm c}, \quad \phi^{\pm}_{\pm n}, \quad \xi^{\pm}_m, \qquad n \in \mathbb{Z}_{\geq 0}, \quad m \in \mathbb{Z}, \tag{1}$$

with $q^{\pm c}$ central and relations expressed in terms of the generating series

$$\xi^{\pm}(z) = \sum_{n\in\mathbb{Z}} \xi^{\pm}_n z^n, \qquad \phi^{\pm}(z) = \sum_{n\geq 0} \phi^{\pm}_{\pm n} z^{\pm n} \tag{2}$$

as follows. Let $\mathcal{B} = \mathbb{C}[[h]]$ be ring formal power series in h, where $q = e^{h/2}$. Let

$$g(z) = \frac{q^2 z - 1}{z - q^2} \quad \in \mathcal{B}[[z]] \tag{3}$$

be an element of $\mathcal{B}$ with coefficients formal power series in z. (Throughout this paper the notation $f(z)$ will be used to denote functions in $\mathcal{B}[[z]]$, $f(z^{-1}) \in \mathcal{B}[[z^{-1}]]$, etc..)

The relations between the generating series (2) are the following formal power series relations:

$$\begin{aligned}
&\phi^+(0)\phi^-(0) = \phi^-(0)\phi^+(0) = 1, \\
&\left[\phi^{\pm}(z), \phi^{\pm}(w)\right] = 0, \\
&\phi^+(z)\phi^-(w) = \frac{g(\frac{z}{w}q^{-c})}{g(\frac{z}{w}q^{c})}\phi^-(w)\phi^+(z) \\
&\phi^{\pm}(z)\xi^{\pm}(w) = g\left(\left(\tfrac{z}{w}\right)^{\pm 1}\right)\xi^{\pm}(w)\phi^{\pm}(z) \\
&\phi^{\pm}(z)\xi^{\mp}(w) = g\left(\left(\tfrac{z}{w}q^c\right)^{\pm 1}\right)^{-1}\xi^{\mp}(w)\phi^{\pm}(z) \\
&(z - q^{\pm 2}w)\xi^{\pm}(z)\xi^{\pm}(w) = (q^{\pm 2}z - w)\xi^{\pm}(w)\xi^{\pm}(z) \\
&\left[\xi^+(z), \xi^-(w)\right] = (q - q^{-1})\left(\delta(\tfrac{z}{w}q^{-c})\phi^-(w) - \delta(\tfrac{z}{w}q^{c})\phi^+(z)\right).
\end{aligned} \tag{4}$$

Here, the function $\delta(z)$ is a formal series,

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

It can also be regarded as a distribution which acts on the space of functions $f(z)$ regular at $z = 1$.

2.2. **Hopf algebra structure.** As an algebra, Drinfeld's algebra (4) is isomorphic to the Drinfeld-Jimbo $U_q(\widehat{sl}_2)$ [Dr, J, B]. However it can be endowed with the Hopf

algebra structure of [Dr2] which is different from that found in [Dr, J]. In this paper, the notation $U_q(\widehat{sl}_2)$ refers to Drinfeld's algebra as a Hopf algebra. **Co-product:**

$$\begin{aligned}\Delta(q^c) &= q^c \otimes q^c,\\ \Delta(\xi^+(z)) &= \xi^+(z)\otimes 1 + \phi^+(z)\otimes \xi^+(zq^{c_1}),\\ \Delta(\xi^-(z)) &= 1\otimes \xi^-(z) + \xi^-(zq^{c_2})\otimes \phi^-(z),\\ \Delta(\phi^+(z)) &= \phi^+(z)\otimes \phi^+(zq^{c_1}),\\ \Delta(\phi^-(z)) &= \phi^-(zq^{c_2})\otimes \phi^-(z).\end{aligned} \tag{5}$$

Here, c_i, $i = 1, 2$ is the value of the central charge acting on the i-th factor in the tensor product.

Co-unit:

$$\varepsilon(q^c) = 1,\ \varepsilon(\phi^\pm(z)) = 1,\ \varepsilon(\xi^\pm(z)) = 0. \tag{6}$$

Antipode:

$$\begin{aligned}S(q^c) &= q^{-c},\\ S(\phi^\pm(z)) &= \phi^\pm(zq^{-c})^{-1},\\ S(\xi^+(z)) &= -\phi^+(zq^{-c})^{-1}\xi^+(zq^{-c}),\\ S(\xi^-(z)) &= -\xi^-(zq^{-c})\phi^-(zq^{-c})^{-1}.\end{aligned} \tag{7}$$

These satisfy the condition $m\circ(S\otimes\mathrm{id})\circ\Delta = \varepsilon = m\circ(\mathrm{id}\otimes S)\circ\Delta$, where $m(x\otimes y) = xy$.

Remark 2.1 *Normal ordering*: Let $\mathcal{O}$ be the category of highest weight representations, where the positive modes $\phi^+_n, \xi^\pm_n, \phi^-_{-n}, n > 0$ act nilpotently to the right. The product of two generating functions in $U_q(\widehat{sl}_2)$ is well defined when it is normally ordered, i.e. all positive modes on the right and the negative on the left. Then in the category $\mathcal{O}$ normally ordered products of generating functions act as Laurent series.

The relations above describe a Hopf algebra structure in the category $\mathcal{O}$. The antipode should have the property $\langle xv, w\rangle = \langle v, S(x)w\rangle$, $x \in U_q(\widehat{sl}_2), v \in M^*, w \in M$, where $M \in \mathcal{O}$, $M^* \in \mathcal{O}^*$ the right dual to M, and $\mathcal{O}^*$ is the category of lowest weight modules. Therefore acting by S on a normally ordered product of generating functions gives an anti-normally ordered product which acts in $\mathcal{O}^*$ as Laurent series.

2.3. **The R-matrix.** Let $V = \mathrm{End}(\mathbb{C}^2)$ be the two dimensional representation of sl_2 with basis v_1, v_2. Consider the two-dimensional evaluation representation of $U_q(\widehat{sl}_2)$, $V_z = V\otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$. The map

$$\pi_z : U_q(\widehat{sl}_2)|_{c=0} \to V[z, z^{-1}]$$

is defined by [DI]

$$\begin{aligned}&\xi^\pm(w) \mapsto (q - q^{-1})\ \delta(\tfrac{w}{z})\ \sigma^\pm && \in V[z, z^{-1}][[w, w^{-1}]]\\ &\phi^+(w) \mapsto \begin{pmatrix} d(q^2\frac{w}{z}) & 0\\ 0 & d(\frac{w}{z})^{-1}\end{pmatrix} && \in V[z^{-1}][[w]],\\ &\phi^-(w) \mapsto \begin{pmatrix} d(\frac{z}{w})^{-1} & 0\\ 0 & d(q^2\frac{z}{w})\end{pmatrix} && \in V[z][[w^{-1}]],\end{aligned} \tag{8}$$

where

$$d(z)=\frac{1-z}{q-q^{-1}z} \tag{9}$$

and $\sigma^{\pm}$ are the Pauli matrices. Here, the notation $\mathbb{C}[z][[w]]$ indicates formal power series in w with coefficients polynomials in z, etc.

Proposition 2.1. *There is a unique, up to a scalar multiple, 4×4 matrix $\widetilde{R}(z)$ with diagonal entries in $\mathcal{B}[[z]]$ and off diagonal elements in $\mathcal{B}[[z,z^{-1}]]$ which satisfies the intertwining relation*

$$\widetilde{R}(z/z')(\pi_z\otimes\pi_{z'})\Delta(x)=(\pi_z\otimes\pi_{z'})\Delta'(x)\widetilde{R}(z/z'),\ x\in U_q(\widehat{sl}_2). \tag{10}$$

This unique solution is $\widetilde{R}(z)=R_{12}^{-1}(z)$, where

$$R_{12}(z)=f(z)\begin{pmatrix}1&&&\\&d(z)&0&\\&\gamma(z)\delta(z)&d(q^2z)&\\&&&1\end{pmatrix}, \tag{11}$$

where $\gamma(z)=(q-q^{-1})d(z)=\frac{(q-q^{-1})(1-z)}{q-q^{-1}z}$.

Remark 2.2 Here, $(1-z)^n\delta(z)$, $n\in\mathbb{Z}_{\geq 0}$ can be thought of as a distribution acting on the space of functions with poles of order less than or equal to n.

The proof of proposition (2.1) is provided in the appendix.

The matrix R will be used in the next section to define the algebra $A(R)$. The scalar factor $f(z)$ is uniquely determined from the requirement that the quantum determinant is central (see next section), which gives

$$f(z)=q^{1/2}(1-z)\widetilde{g}(z)^2\qquad\in\mathcal{B}[[z]], \tag{12}$$

where

$$\widetilde{g}(z)=\frac{(q^4z;q^4)_\infty}{(q^2z;q^4)_\infty},\qquad (z,q)_n=\prod_{j=0}^{n}(1-zq^j).$$

The function $f(z)$ in (12) is the unique solution to the difference equation

$$f(z)f(q^2z)=d(q^2z)^{-1},\qquad f(z)\in\mathcal{B}[[z]]. \tag{13}$$

With this choice of $f(z)$, the matrix R is equal to the evaluation of the universal R-matrix [R].

2.4. **Extension of Drinfeld's algebra.** For what follows it will be useful to describe a slightly extended version of Drinfeld's algebra, $\overline{U}_q(\widehat{sl}_2)$ [FR]. Consider the generators $\alpha^{\pm}_{\pm n}$,

$$\alpha^{\pm}(z)=\sum_{n\geq 0}\alpha^{\pm}_{\pm n}z^{\pm n},$$

such that

$$\phi^{\pm}(z)=\alpha^{\pm}(z)^{-1}\alpha^{\pm}(q^2z)^{-1}. \tag{14}$$

The algebra $\overline{U}_q(\widehat{sl}_2)$ is generated by $\xi^{\pm}(z), \alpha^{\pm}(z), q^c$ with relations as in (4) and

$$\begin{aligned}
&\left[\alpha^{\pm}(z), \alpha^{\pm}(w)\right] = 0, \\
&\alpha^{+}(z)\xi^{+}(w) = d(\tfrac{z}{w})^{-1}\xi^{+}(w)\alpha^{+}(z), \\
&\alpha^{-}(z)\xi^{-}(w) = d(q^2\tfrac{w}{z})^{-1}\xi^{-}(w)\alpha^{-}(z), \\
&\alpha^{+}(z)\xi^{-}(w) = d(\tfrac{z}{w}q^c)\xi^{-}(w)\alpha^{+}(z), \\
&\alpha^{-}(z)\xi^{+}(w) = d(\tfrac{w}{z}q^{-c+2})\xi^{+}(w)\alpha^{-}(z), \\
&\alpha^{+}(z)\alpha^{-}(w) = \frac{f(zq^c/w)}{f(zq^{-c}/w)}\alpha^{-}(w)\alpha^{+}(z).
\end{aligned} \tag{15}$$

The Hopf algebra structure extends to $\overline{U}_q(\widehat{sl}_2)$:

$$\begin{aligned}
&\Delta(\alpha^{+}(z)) = \alpha^{+}(z) \otimes \alpha^{+}(zq^{c_1}), \\
&\Delta(\alpha^{-}(z)) = \alpha^{-}(zq^{c_2}) \otimes \alpha^{-}(z), \\
&S(\alpha^{\pm}(z)) = \alpha^{\pm}(zq^{-c})^{-1}, \\
&\varepsilon(\alpha^{\pm}(z)) = 1.
\end{aligned} \tag{16}$$

The two dimensional evaluation representation of $\alpha^{\pm}(z)$ is obtained by solving the difference equation (14). The result is

$$\begin{aligned}
\pi_z\alpha^{+}(w) &= f(\tfrac{w}{z}) \begin{pmatrix} 1 & 0 \\ 0 & d(\frac{w}{z}) \end{pmatrix}, \\
\pi_z\alpha^{-}(w) &= \frac{1}{f(\frac{z}{w})} \begin{pmatrix} 1 & 0 \\ 0 & d(q^2\frac{z}{w})^{-1} \end{pmatrix}.
\end{aligned} \tag{17}$$

3. The quantum current algebra $A(R)$

We define the Hopf algebra $A(R)$ using the same methods introduced in [RSTS].

Consider the algebra generated by the coefficients of $L^{\pm}_{ij}(z)$, with $i, j = 1, 2$, $L^{\pm}_{ii}(z) \in A(R)[[z^{\pm 1}]]$, and the off diagonal elements in $A(R)[[z, z^{-1}]]$. The matrix L^{+} (L^{-}) is lower (upper) triangular.

The determining relations are

$$\begin{aligned}
R_{12}(\tfrac{z}{w})L^{\pm}_1(z)L^{\pm}_2(w) &= L^{\pm}_2(w)L^{\pm}_1(z)R_{12}(\tfrac{z}{w}), \\
R_{12}(\tfrac{z}{w}q^{-c})L^{+}_1(z)L^{-}_2(w) &= L^{-}_2(w)L^{+}_1(z)R_{12}(\tfrac{z}{w}q^{c}).
\end{aligned} \tag{18}$$

Here the matrix $R(z)$ is the same as in (11) with $f(z)$ as in (12). We choose $f(z)$ by requiring that the quantum determinants

$$\mathcal{D}^{\pm}(z) = L^{\pm}_{11}(z)L^{\pm}_{22}(zq^{-2})$$

are central. The choice of the shift q^{-2} in the definition of $\mathcal{D}^{\pm}(z)$ is motivated by the identification of the evaluation of $L^{+}(z)$ with $R(z)$ (see below).

Finally, the algebra $A(R)$ is defined as the quotient of the algebra generated by $L^{\pm}_{ij}(z)$ with relations (18) by the relation $\mathcal{D}^{\pm}(z) = 1$.

Remark 3.1 This construction is motivated by the following considerations. Suppose there exists a universal R-matrix for $\overline{U}_q(\widehat{sl}_2)$ such that $L^{+}(z) = (\mathrm{id} \otimes \pi_z)\mathcal{R}$, $L^{-}(z) = (\mathrm{id} \otimes \pi_z)\mathcal{R}^{-1}_{21}$ and $R(z/w) = (\pi_z \otimes \pi_w)\mathcal{R}$ [R]. Then the relations (18) follow from the Yang-Baxter relation for $\mathcal{R}$.

The algebra $A(R)$ is a Hopf algebra with comultiplication

$$\Delta L^+(z) = L^+(z)\dot{\otimes}L^+(zq^{c_1}), \quad \Delta L^-(z) = L^-(zq^{c_2})\dot{\otimes}L^-(z) \tag{19}$$

which is easily seen to be an algebra homomorphism of (18). (Here the notation $\dot{\otimes}$ indicates matrix multiplication in V and tensor product in $A(R)$.) The co-unit acts as $\varepsilon(L^\pm) = 1$, and the action of the antipode is

$$S(L^\pm(z)) = L^\pm(zq^{-c})^{-1}.$$

Remark 3.2 Since $\mathcal{D}^\pm(z) = 1$, $L^\pm_{ii}(z)^{-1} \in A(R)$. The matrices $S(L^\pm(z)) = L^\pm(z)^{-1}$ are well defined in the category $\mathcal{O}^*$, as they should be according to remark (2.1.

Define the map

$$\mu : A(R) \to \overline{U}_q(\widehat{sl}_2)$$

as

$$\begin{aligned} L^+(z) &\mapsto \begin{pmatrix} \alpha^+(z) & 0 \\ \xi^+(z)\alpha^+(z) & \alpha^+(q^2z)^{-1} \end{pmatrix}, \\ L^-(z) &\mapsto \begin{pmatrix} \alpha^-(z) & -\alpha^-(z)\xi^-(z) \\ 0 & \alpha^-(q^2z)^{-1} \end{pmatrix}. \end{aligned}$$

Proposition 3.1. *The map μ is a Hopf algebra isomorphism.*

The proof is by direct calculation.

By using the evaluation representation for the Drinfeld generators, it is easily shown that

$$\pi_w L^+(z) = R(z/w), \qquad \pi_w L^-(z) = R_{21}^{-1}(w/z).$$

(Here, we use the notation $(\pi_w L^+(z)_{ij})_{kl} = R_{ij,kl}(z/w)$ where the action of R on the basis vectors of V is $R(z)v_i \otimes v_j = \sum_{k,l=1}^2 R_{ij,lk}(z)v_k \otimes v_l$.

The relations (18) in the evaluation representation are therefore equivalent to the Yang-Baxter equation for R:

$$R_{12}(z)R_{13}(wz)R_{23}(w) = R_{23}(w)R_{13}(wz)R_{12}(z). \tag{20}$$

The intertwining relation (10) with $x = L^\pm(w)_{ij}$ is again a simple consequence of the Yang-Baxter relations (20), with the co-product as in (19).

Note that this Hopf algebra isomorphism provides a trivial proof that Δ is an algebra homomorphism of Drinfeld's algebra (*c.f.* [DI]).

4. The Poisson algebra limit

The Poisson limit $h \to 0$ of Drinfeld's algebra is obtained by keeping $p = q^c$ and the generators $\xi^\pm$ and $\phi^\pm$ or $\alpha^\pm$ fixed. In this limit the algebra $\overline{U}_q(\widehat{sl}_2)$ becomes a Poisson algebra with

$$\{a, b\} = \lim_{h\to 0} \frac{a\cdot b - b\cdot a}{h}, \quad a, b \in \overline{U}_q(\widehat{sl}_2)$$

Explicitly,

$$\phi^\pm(z) = \alpha^\pm(z)^{-2}$$

and

$$
\begin{aligned}
&\{\alpha^\pm(z),\alpha^\pm(w)\} = 0,\\
&\{\xi^+(z),\xi^-(w)\} = \left(\delta(\tfrac{z}{w}p^{-1})\phi^-(w) - \delta(\tfrac{z}{w}p)\phi^+(z)\right),\\
&\{\xi^\pm(z),\xi^\pm(w)\} = \pm\frac{1}{2}\left(\lambda(\tfrac{w}{z}) - \lambda(\tfrac{z}{w})\right)\xi^\pm(z)\xi^\pm(w),\\
&\{\alpha^+(z),\alpha^-(w)\} = \frac{1}{4}\left(\lambda(\tfrac{z}{w}p) - \lambda(\tfrac{z}{w}p^{-1})\right)\alpha^+(z)\alpha^-(w),\\
&\{\alpha^\pm(z),\alpha^\pm(w)\} = -\frac{1}{2}\lambda((\tfrac{z}{w})^{\pm 1})\alpha^\pm(z)\xi^\pm(w),\\
&\{\alpha^\pm(z),\xi^\mp(w)\} = \frac{1}{2}\lambda((\tfrac{z}{w}p)^{\pm 1})\alpha^\pm(z)\xi^\mp(w).
\end{aligned}
\tag{21}
$$

Again, these are formal power series relations with

$$
\lambda(z) = \frac{1}{4}\,\frac{1+z}{1-z}.
$$

The algebra $A(R)$ in this limit is a Poisson algebra generated by the elements $\mathcal{L}^\pm_{ij}(z)$ as above, and the quantum determinant is the usual determinant. The relations (18) become, in this limit,

$$
\begin{aligned}
&\left\{\mathcal{L}_1^\pm(z),\mathcal{L}_2^\pm(w)\right\} = \left[\mathcal{L}_1^\pm(z)\mathcal{L}_2^\pm(w), r(\tfrac{z}{w})\right],\\
&\left\{\mathcal{L}_1^+(z),\mathcal{L}_2^-(w)\right\} = \mathcal{L}_1^+(z)\mathcal{L}_2^-(w)r(\tfrac{z}{w}p^{-1}) - r(\tfrac{z}{w}p)\mathcal{L}_1^+(z)\mathcal{L}_2^-(w).
\end{aligned}
\tag{22}
$$

The classical r-matrix is obtained from $R(z)$ by taking the limit $h \to 0$ of the elements of R. We find

$$
R(z) = 1 + h\ r(z) + \mathcal{O}(h^2). \tag{23}
$$

Here we consider the off diagonal element of R to be in $\mathcal{B}[[z,z^{-1}]]$. This means that the coefficient in front of the δ function in r is the coefficient of h in the expansion of $(q-q^{-1})f(z)/d(z)$, which is 1. Thus, the classical r-matrix is

$$
r(z) = \begin{pmatrix} \lambda(z) & 0 & 0 & 0 \\ 0 & -\lambda(z) & 0 & 0 \\ 0 & \delta(z) & -\lambda(z) & 0 \\ 0 & 0 & 0 & \lambda(z) \end{pmatrix}. \tag{24}
$$

The isomorphism μ gives, in this limit,

$$
\mathcal{L}^+ = \begin{pmatrix} \alpha^+(z) & 0 \\ \alpha^+(z)x^+(z) & \alpha^+(z)^{-1} \end{pmatrix}, \quad \mathcal{L}^- = \begin{pmatrix} \alpha^-(z) & \alpha^-(z)x^-(z) \\ 0 & \alpha^-(z)^{-1} \end{pmatrix}. \tag{25}
$$

Since all the operators commute there is no problem with normal ordering in this limit.

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Appendix A. The intertwining relation

The following is a proof of Proposition (2.1).

Let

$$\widetilde{R}(z) = \begin{pmatrix} a_1(z) & 0 & 0 & 0 \\ 0 & b_1(z) & \gamma_1\delta(z) & 0 \\ 0 & \gamma_2\delta(z) & b_2(z) & 0 \\ 0 & 0 & 0 & a_2(z) \end{pmatrix},$$

with the diagonal elements in $\mathbb{C}[[z]]$ and the off-diagonal elements in $\mathbb{C}[[z, z^{-1}]]$. Then, up to a scalar multiple, the unique solution to the intertwining relation (10) is $R_{12}^{-1}(z)$.

The proof consists of using the evaluation representation (8) to explicitly compute

$$\widetilde{R}(z/z')(\pi_z \otimes \pi_{z'})\Delta(\xi^{\pm}(w)) = (\pi_z \otimes \pi_{z'})\Delta'(\xi^{\pm}(w))\widetilde{R}(z/z').$$

Writing out these relations, it is immediately apparant that the off-diagonal terms of the matrix $\widetilde{R}$ must be proportional to the δ function if they are nonzero. Cancelling an overall factor of $\delta(w/z)$[1] and setting $z' = 1$, this results in eight relations between formal power series in z and z^{-1}:

$$a_1(z)d(q^2z^{-1}) = b_1(z) + d(q^2z)\gamma_2(z)\delta(z), \tag{26}$$
$$a_1(z) = \gamma_1(z)\delta(z) + d(q^2z)b_2(z), \tag{27}$$
$$b_1(z) + \gamma_1(z)\delta(z)/d(z^{-1}) = a_2(z)/d(z), \tag{28}$$
$$\gamma_2(z)\delta(z) + b_2(z)/d(z^{-1}) = a_2(z), \tag{29}$$
$$b_1(z) + \gamma_1(z)\delta(z)/d(z^{-1}) = a_1(z)/d(z), \tag{30}$$
$$\gamma_2(z)\delta(z) + b_2(z)/d(z^{-1}) = a_1(z), \tag{31}$$
$$a_2(z)d(q^2z^{-1}) = b_1(z) + d(q^2z)\gamma_2(z)\delta(z), \tag{32}$$
$$a_2(z) = \gamma_1(z)\delta(z) + b_2(z)d(q^2z). \tag{33}$$

The first four relations come from considering $\Delta(xi^+)$ in (10) and the last four from $\Delta(\xi^-)$.

From (27), $\gamma_1(z) = 0$ since all other terms are in $\mathbb{C}[[z]]$ and there is nothing to cancel terms in $\mathbb{C}[[z^{-1}]]$ coming from $\delta(z)$. Therefore, from (28), $b_1(z) = \frac{a_2(z)}{d(z)}$, and from (30), $b_1(z) = \frac{a_1(z)}{d(z)}$, so $a_2 = a_1$. From (33), $a_1(z) = b_2(z)d(q^2z)$.

As formal power series,

$$\frac{1}{1-z^{-1}} = \sum_{n\le 0} z^n = \delta(z) - \frac{z}{1-z},$$

and therefore

$$d^{-1}(z^{-1}) = d(q^2z) + (q - q^{-1})\delta(z).$$

From (29),

$$\gamma_2(z)\delta(z) = (q^{-1} - q)b_2(z)\delta(z).$$

These relations determine R only up to a scalar multiple. Let

$$a_1(z) = \frac{a(z)}{1-z}.$$

[1]In the sense the coefficient in front of $\delta(w/z)$, since it is independent of w, must be zero.

Then

$$b_1(z) = \frac{a(z)}{(1-z)d(z)}, \quad b_2(z) = \frac{a(z)}{(1-z)d(q^2z)}, \quad \gamma_2(z) = \frac{(q^{-1}-q)a(z)}{(1-z)d(q^2z)}.$$

Thus $\widetilde{R}$ is identified with the matrix $R^{-1}(z)$

$$R_{12}^{-1}(z) = q^{-1/2}\widetilde{g}(z)^{-2} \begin{bmatrix} \frac{1}{1-z} & & & \\ & \frac{q-q^{-1}z}{(1-z)^2} & 0 & \\ & \gamma'(z)\delta(z) & \frac{1}{q^{-1}-qz} & \\ & & & \frac{1}{1-z} \end{bmatrix}, \tag{34}$$

with

$$\gamma'(z) = \frac{q^{-1}-q}{(1-z)d(q^2z)} = \frac{q^{-1}-q}{q^{-1}-qz},$$

with $a(z) = q^{-1/2}\widetilde{g}(z)^{-2}$.

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